The Etale Site

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We will follow [Mil] pretty closely.

1 Sites

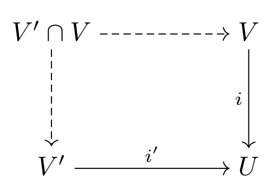
We are going to categorify our idea of topology. The idea is to consider the topology as *only* the open sets, as a lattice, and not worry about the points that are contained in it. This perspective is ubiquetous in AG and explains their abuse of the word point.

Example. *We start with a topological space $(X, \mathcal{T})$. To this there is a naturally associated category*

$$Open(X) := \begin{cases} objects: & open\ sets \\ morphisms: & inclusion\ maps \end{cases}$$

This has the important categorical property of all pullbacks existing

is there a word for this?



If you include into V and V' then you must include into their intersection.

A site is a category $\mathscr{C}$ along with the data of "coverings". That is for every object $c \in \mathscr{C}$ there is a set of "coverings", a collection of collections of maps $Cov(c) = \{(U_i \to c)^j_{i \in I}\}_{j \in J}$ which satisfy the normal topological properties of coverings

- The identity map is a covering.

- For any covering $(U_i \to c)_i$ and any morphism $V \to c$ (in $\mathscr{C}$) the fiber products / pullbacks exists and

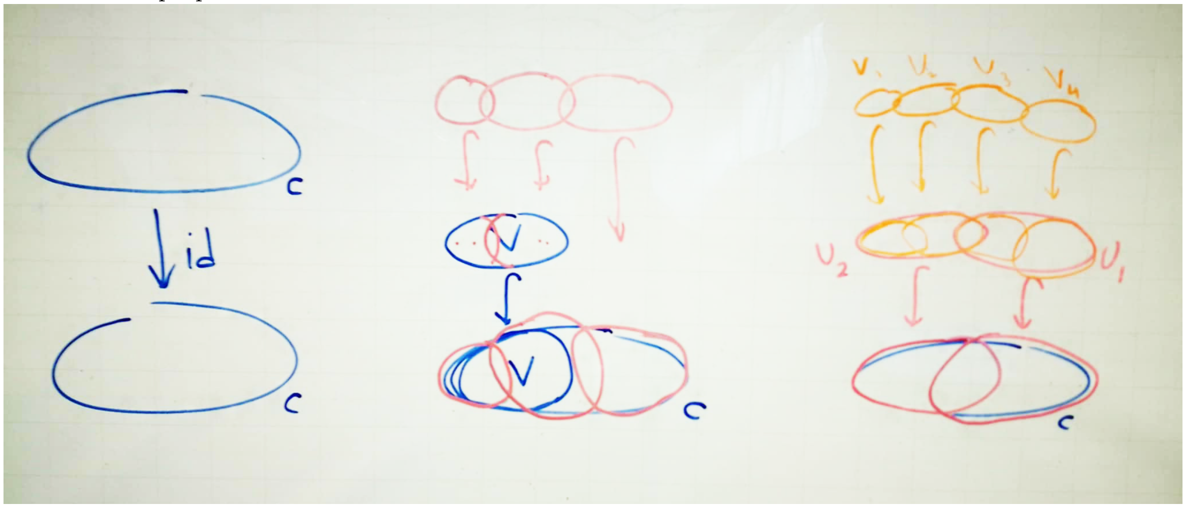
$$(U_i \times_c V \to V)$$

is itself a covering of V, i.e. it is an element of $Cov(V)$

- If $(U_i \to c)_i$ is a covering and for each $i \in I$ $(U_{i,j} \to U_i)_j$ is a covering of U_i then $(V_{i,j} \to c)_{i,j}$ is a covering of c

Example. *In the case of the category Open(X), then normal topological coverings suffice. I will illustrate the properties*

> when is a site actually given by a topological space?



Example. *If X is a scheme then we have the (small) Zariski site X_{zar} given by treating it as a topological space.*

2 Sheaves on Sites

Presheaves of some things on a site $(\mathscr{C}, J)$ are defined as usual as functors

$$\mathscr{F} : \mathscr{C}^{op} \to \text{ Category of those things}$$

such a presheaf $\mathscr{F}$ is in addition a sheaf if for every $c \in \mathscr{C}$ and every covering $(U_i \to c)_i$ $\mathscr{F}(c)$ equalizes the diagram

$$\prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_c U_j)$$

We denote $Sh(\mathscr{C}, \mathscr{D})$ to be the category of sheaves in $\mathscr{D}$ on the site $\mathscr{C}$, where the target category is clear from context we simply write $Sh(\mathscr{C})$. Categories of the form $Sh(\mathscr{C}, \text{Set})$ are called Topoi if they are abelian they are trivial.

Example. *Any sheaf on a topological space X becomes a sheaf on Open(X) in the obvious way.*

Example (Hom)**.** *In particular if X, Y are topological spaces then there is a sheaf (of sets) of continuous Y valued functions on X, namely $\text{Hom}_{\text{Top}}(-, Y)$, which indeed lifts to a sheaf on the site of Open(X).*

More relevant (and fixed from now on) is $Sh(\mathscr{C}, \mathbb{Z}\text{-modules})$ or sheaves of abelian groups. This category has two very nice properties it is an abelian category and moreover has enough injectives. This allows us to define sheaf cohomology as a right derived functor. In detail let

$$\Gamma(-,-) : Sh(\mathscr{C}) \times \mathscr{C} \to \mathrm{Ab}$$

$$\Gamma(\mathscr{F}, U) = \mathscr{F}(U)$$

Then the sheaf cohomology of an object $U \in \mathscr{C}$ is defined to be the right derived functor of $\Gamma(-, U)$. In particular

$$H^r(U, \mathscr{F}) \quad := \quad H^r(0 \to \Gamma(\mathcal{I}_0, U) \to \Gamma(\mathcal{I}_1, U) \to \cdots)$$

where

$$0 \to \mathscr{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

is an injective resolution of $\mathscr{F}$.

Example (Simplicial)**.** *It is a theorem that under some conditions on the topological space X (paracompact, locally contractable) that for any abelian group G we have that*

$$H^i(X, \underline{G}) \cong H^i_{\text{sing}}(X, A)$$

where $X \in Open(X)$ and $\underline{G}$ is the constant sheaf on this site assigning G to every open.

`https://public.websites.umich.edu/~mmustata/SingSheafcoho.pdf`
`https://arxiv.org/pdf/1602.06674`

3 The Etale Topology

3.1 Zariski Bad

If you want a cohomology theory for varieties there are two things to try, do the simplicial thing directily, which Oliver claims is not very nice, or useing the above theorem to motivate using sheaf cohomology. This doesnt work either unfortunately. A topological space is irreducible if it cannot be written as the union of two proper closed subsets, or equivilently any two non-empty opens have a non-empty intersection.

Theorem. *If X is an irreducible topological space and $\mathscr{F}$ is a constant sheaf then*

$$H^r(X, \mathscr{F}) = 0, \quad r > 0$$

Proof. See [Mil80, Thm 1.1]

This clearly applies to varieties and so we might start to worry, the zariski topology on a variety seems to always have trivial cohomology, and this generalises to schemes which are just built out of these things. So basically cohomology theories on these sorts of things are going to be degenerate. So the Zariski site is not the right topology for cohomology of varieties.

3.2 Etale Good

Now we are thinking of topological spaces in a point free way and we see that what we really need is to specify covers. But if we define a class of morphisms that are preserved under pullback and the other cover operations we can think of them as a "sub-topology" or a sub cover. So this is what we will do we will introduce a class of morphisms and then take covers to be morphisms of schemes satisfying that property.

3.2.1 Commutative Algebra

Fix X, Y two schemes. To define a property of a morphism $\varphi : Y \to X$ we will do what we always do; we will define a property of maps of rings and then ask that the morphism of schemes satisfies this property on the stalks or some affine opens.

A morphism of rings $A \to B$ is ***flat*** if the functor

$$M \mapsto M \otimes_A B$$

is exact (recall that a morphism of rings is the same as an A algebra). Then φ is flat if the induced map on stalks is

$$\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$$

is flat for every $y \in Y$. Or equivilently if for all affine opens the map induced on the sections of those opens is flat.

A local homomorphism of local rings $f : A \to B$ is ***unramified*** if $B/f(\mathfrak{m}_A)B$ is a finite seperable extension of $A/\mathfrak{m}_A$. Then φ is unramified if it is of finite type and the induced map

$$\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$$

is unramified for every $y \in Y$.

A morphism is ***etale*** if it is both flat and unramified.

Example. *Any projective module is flat, in particular free modules are flat. Therefore spec'ing these morphisms give flat morphisms of schemes.*

If L/K is an extension of number fields with respective rings of integers $\mathcal{O}_K, \mathcal{O}_L$, then the inclusion $K \to L$ induces an inclusion $\mathcal{O}_K \to \mathcal{O}_L$ which gives a morphism after spec'ing

$$f : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$$

then this is unramified at $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$ iff this prime ideal is unramified in the algebraic number theory sense, which we recall means that the residue fields are seperable and the powers are 1 for each prime appearing in the decomposition of the image of $\mathfrak{p}$ under this map.

This map also turns out to be flat etc. But it requires more commutative algebra.

3.2.2 Geometry

Lets look at what the geometric meaning of this definition is. Consider W, V non-singular algebraic varieties over an algebraically closed field k. Let $\varphi : W \to V$ a regular map.

Theorem. *φ is etale at $x \in W$ iff the induced map*

$$\mathrm{d}\varphi : T_x W \to T_{\varphi(x)} V$$

is an isomorphism. φ is etale iff it is etale at every point.

Proof. [Mil80, Prop 2.9]

this condition is reminiscent of the inverse mapping theorem for smooth things and so we might hope that there is some condition on the Jacobian of the map, whatever that means. We have such a condition

Theorem (Stacks Project Lemma 29.36.15)**.** *Let*

$$f : \mathrm{Spec}\,\frac{R[x_1, ..., x_n]}{(f_1, ..., f_n)} \to \mathrm{Spec}\,R$$

then this map is etale at $\mathfrak{q} \subseteq \frac{R[x_1,...,x_n]}{(f_1,...,f_n)}$ (prime) if

$$det(\frac{\partial f_i}{\partial x_j})_{i,j} \notin \mathfrak{q}$$

recall that these are just the conditions for the inverse mapping theorem, so etale maps are reminiscent of local diffeomorphisms.

Example. *Consider the map*

$$\mathbb{A}^1_k \to \mathbb{A}^1_k$$
$$\operatorname{Spec} k[x] \to \operatorname{Spec} k[s]$$

That is induced by spec'ing the following inclusion

$$k[s] \to k[s][x]/(s - x^n) \cong k[x]$$
$$s \mapsto x^n$$

Now the jacobian is

$$\frac{\partial(s^n - x)}{\partial s} = ns^{n-1}$$

and we need to identify which prime ideals of $k[x][s]/(s - x^n)$ contain the polynomial ns^{n-1}, or equivilently by pulling back along the above isomorphism the prime ideals of $k[x]$ which include $n(x^n)^{n-1} = nx^{n^2-n}$.

Clearly if $char(k)|n$ then this map is clearly never etale, as $nx^{n^2-n} = 0 \in (0)$ and hence contained in every prime. One can also see that it is not etale at (x) because clearly $nx^{n^2-n} \in (x)$. It is etale everywhere else because (x) is not contained in any other prime ideal.

Example (Field Extensions)**.** *Consider a field extension*

$$\mathbb{Q} \to \mathbb{Q}[3\sqrt{17}] \cong \frac{\mathbb{Q}[x]}{x^3 - 17}$$

then we want to look at when the spec of this map is etale. The Jacobian is just

$$\frac{\partial(x^3 - 17)}{\partial x} = 3x^2$$

Moreover $\mathbb{Q}[x]/(x^3 - 17)$ is a field, hence its only prime ideal is (0). So this is etale for because in $\mathbb{Q}[x]/(x^3 - 17)$, $3x^2 \neq 0$

3.3 The Etale Sites

Given a scheme X we can define two etale sites. We will probably only need one in this seminar. There is the small etale site

$$X_{et} \quad := \begin{cases} \textit{objects}: & \text{Etale morphisms } U \to X \\ \textit{morphisms}: & \text{Morphisms of schemes over } X, \;\; U \to V \\ \textit{coverings}: & \text{Surjective families of etale maps } (U_i \to U)_i \end{cases}$$

and the big etale site

$$X_{Et} \quad := \begin{cases} \textit{objects}: & \text{Schemes over } X \\ \textit{morphisms}: & \text{Morphisms of schemes over } X \\ \textit{coverings}: & \text{Surjective families of etale maps } (U_i \to U)_i \end{cases}$$

The difference is that the small etale site is somehow only schemes that are etale over X, not all schemes over X. We think of these as "etale open subsets". For $X = \operatorname{Spec} \mathbb{Z}$ the big etale site is a topology on the category of all schemes. We can now define the etale cohomology of a scheme (and a sheaf on the etale site), it is simply

$$H^\bullet(X_{et}, \mathscr{F})$$

by analogy with our singular cohomology example, if $\mathscr{F}$ is some constant sheaf then this is the etale cohomology of X with those coefficients.

4 Sheaves on the Etale Site

Example ([?], 2.1.10)**.** *A sheaf of groupoids on* $\operatorname{Spec}\mathbb{Z}_{Et}$ *is called a stack.*

So one can think about categories as topological spaces where all the morphisms are the covers and a terminal object is sort of the "biggest open set", of course you need completness and things. From this perspective the category of schemes is just a topological space, the set is $\operatorname{Spec}\mathbb{Z}$ and the opens are all the maps. Then schemes are (particular) sheaves of sets on this topological space.

Example. *If* Z *is a scheme over* X *then the functor*

$$\operatorname{Hom}_X(-, Z)$$

defines a sheaf of sets on X_{et}*.*

If Z *is also a group scheme then this sheaf is a sheaf of groups.*

Example. *Let* μ_n *be the scheme defined by* $T^n - 1 = 0$ *i.e.*

$$\mu_n \quad := \quad \operatorname{Spec} R[T]/(T^n - 1)$$

Then $\mu_n(U) \quad := \quad \operatorname{Hom}_X(U, \mu_n)$ *is the group of* $n-th$ *roots of unity in* $\mathcal{O}_U(U)$*.*

Example. *Define* Gl_n *to be*

$$\operatorname{Spec} R[T_{ij} : 1 \leq i, j \leq n][Y]/(Y det(T_{ij}) - 1)$$

Then $\operatorname{Gl}_n(U)$ *is the group of* $n \times n$*-matricies with entries in* $\mathcal{O}_U(U)$*.*

Example (Structure Sheaf)**.** *Let* $U \to X$ *be etale. Then we define a sheaf on the etale site of* X *by*

$$\mathcal{O}_X(U) \quad := \quad \mathcal{O}_U(U)$$

References

[Mil] J S Milne. Lectures on etale cohomology.

[Mil80] J. S. Milne. *Étale Cohomology.* Number 33 in Princeton Mathematical Series. Princeton University Press, Princeton, N.J, 1980.